POINT DERIVATIONS IN CERTAIN SUP-NORM ALGEBRAS(1)

BY
S. J. SIDNEY(2)

1. Let A be a closed point-separating subalgebra of $C(X)$ containing the constants, where X is a compact Hausdorff space. M_A will denote the space of multiplicative linear functionals φ on A, and to each such φ we associate its kernel A_φ. The A_φ are precisely the maximal ideals of A.

Under certain hypotheses, it is known that analytic discs can be embedded in M_A. Wermer [W1] showed that if A is a Dirichlet algebra on X, then each Gleason part of A is either a single point or an analytic disc. Hoffman [H] then generalized Wermer's result to logmodular algebras. Finally Lumer [L] observed that the conclusion is really "local": if φ has a unique representing measure on X, then the part for A containing φ consists either of φ alone or of an analytic disc.

Our objective in this paper is to take the weakest of these possible hypotheses, namely Lumer's, and show that in a broader sense the analytic disc at φ, if there is one, really does account for all the analytic structure at φ. Specifically, we show that all the bounded derivations and higher "derivatives" of A at φ are just differentiations with respect to the analytic structure of the analytic disc.

2. All measures will be regular Borel measures on X; they will be nonnegative real-valued unless they are called complex, in which case they will be complex-valued.

Let $\varphi \in M_A$. Then there is a measure μ representing φ, i.e., $\varphi(f) = \int f \, d\mu$ for all $f \in A$. If there is only one such measure, we will say that φ satisfies condition (U) (for unique). Lumer [L] has shown that condition (U) guarantees the validity of essentially all the logmodular theory of Hoffman's paper [H, §§4–6] as applied to φ. We use this fact freely in the sequel. We shall require two additional facts based on Lumer's paper.

THEOREM 1. *Suppose $\varphi \in M_A$ and μ is a measure representing φ. Then φ satisfies* (U) *if and only if μ satisfies* (U′) $\int u \, d\mu = \sup \{\operatorname{Re} \varphi(f) : f \in A, \operatorname{Re}(f) \leq u\}$ *for all* $u \in C_R(X)$.

Proof. This result is, even strongly generalized, quite familiar (see, for example,

Received by the editors December 20, 1966 and, in revised form, March 9, 1967.

(1) This work comprises a portion of the author's doctoral dissertation [S], written under the direction of Professors Andrew M. Gleason and Lynn H. Loomis and accepted by Harvard University in June 1966.

(2) Research partially supported by National Science Foundation Graduate Fellowship.

[A]). Clearly (U′)⇒(U), since if ν were a second measure representing φ we would have for all $u \in C_R(X)$

$$\begin{aligned}\int u\,d\mu &= \sup\{\operatorname{Re}\varphi(f) : f\in A, \operatorname{Re}(f) \leqq u\}\\ &= \sup\left\{\int \operatorname{Re}(f)\,d\nu : f\in A, \operatorname{Re}(f) \leqq u\right\}\\ &\leqq \int u\,d\nu \leqq \inf\left\{\int \operatorname{Re}(f)\,d\nu : f\in A, \operatorname{Re}(f) \geqq u\right\}\\ &= \inf\{\operatorname{Re}\varphi(f) : f\in A, \operatorname{Re}(f) \geqq u\} = \int u\,d\mu,\end{aligned}$$

hence $\int u\,d\nu = \int u\,d\mu$.

Conversely, suppose (U′) is false and select $u \in C_R(X)$ such that $\int u\,d\mu > \alpha = \sup\{\operatorname{Re}\varphi(f) : f\in A, \operatorname{Re}(f)\leqq u\}$. Define a (real) linear functional L on the (real) subspace of $C_R(X)$ spanned by Re (A) and u, setting $L(\operatorname{Re}(f)+ru) = \operatorname{Re}\varphi(f)+r\alpha$. It is easily seen that L is a positive functional, hence has norm $L(1)=1$. Then one finds a measure ν representing L, i.e., ν represents φ and $\int u\,d\nu=\alpha$. This last equality shows that $\nu\neq\mu$, hence (U) fails to hold. □

Recall that the relation $\varphi\sim\varphi' \Leftrightarrow \|\varphi-\varphi'\|_{A^*}<2$ is an equivalence relation on points of M_A, and the equivalence classes are the (Gleason) parts for A (see for example [H, §7]). We denote the part containing φ by P_φ. The following fact is not actually necessary, but may make it easier for the reader to justify our use of some of Hoffman's results; combined with the aforementioned validity of §4–6 of Hoffman's paper, it immediately guarantees in addition the validity of §7 of Hoffman's paper for any φ satisfying condition (U).

THEOREM 2. *Suppose φ satisfies condition* (U) *and $\varphi' \in P_\varphi$. Then φ' satisfies condition* (U), *and if μ and μ' are the representing measures for φ and φ' respectively, then μ and μ' are mutually bounded absolutely continuous.*

Proof. In the proof of Theorem 6 in [L], Lumer shows that if μ' is *any* representing measure for φ' then μ' is boundedly absolutely continuous with respect to μ. Suppose now μ_1 and μ_2 are two representing measures for φ'. Choose nonnegative real functions $h_1, h_2 \in L^\infty(\mu)$ with $d\mu_j = h_j\,d\mu$. Then $g=h_1-h_2 \in L^1(\mu)$ and $f\in A \Rightarrow \int fg\,d\mu=0$. Since g is real-valued, $f\in A+\bar{A} \Rightarrow \int fg\,d\mu=0$. By Theorem 6.7 of [H], $g=0$ a.e. (μ), so $\mu_1=\mu_2$. □

3. A derivation of A at φ is a linear functional D on A satisfying the usual product rule for derivatives: $D(fg)=D(f)\cdot\varphi(g)+\varphi(f)\cdot D(g)$. The existence of a nonzero derivation at φ is equivalent to $A_\varphi\neq A_\varphi^2$. [Note: If I is an ideal in A, I^n is the ideal generated by products $f_1\cdots f_n$ with $f_i\in I$ if $n\geqq 1$, and $I^0=A$; also, $\bar{I}$ denotes the closure of I.] Similarly, the properness of the inclusion $A_\varphi^3\subset A_\varphi^2$ may be thought of as signifying a sort of second-derivative phenomenon, etc. In this same vein, the existence of a nonzero continuous derivation at φ is equivalent to $A_\varphi\neq(A_\varphi^2)^-$, and so on.

We can now state the main result of this paper.

THEOREM 3. *Suppose* $\varphi \in M_A$ *satisfies condition* (U). *Then either* $P_\varphi = \{\varphi\}$, *in which case* $A_\varphi = (A_\varphi^2)^-$ *and hence* $A_\varphi = (A_\varphi^n)^-$ *for all* $n \geqq 1$, *or there is a homeomorphism h of the open unit disc D in the complex plane onto* P_φ (*in the* A^* *metric topology*) *such that* (D, h) *is an analytic disc at* φ (*i.e.,* $h(0) = \varphi$ *and* $\hat{f} \circ h$ *is analytic for all* $f \in A$, *where* $\hat{f}$ *is the Gelfand transform of* f). *In the latter case,* $(A_\varphi^n)^- / (A_\varphi^{n+1})^-$ *is* 1-*dimensional and* $(A_\varphi^n)^-$ *consists of those* $f \in A$ *for which* $\hat{f} \circ h$ *vanishes at* 0 *to order at least n, for each* $n \geqq 0$.

The description of P_φ as either a point or a disc is of course contained in the papers of Hoffman and of Lumer. Our contribution is the description of the ideals $(A_\varphi^n)^-$ for $n \geqq 2$.

Theorem 3 is proved in the next section. In that section φ will denote a multiplicative linear functional satisfying condition (U) and μ will denote its representing measure. Many of the arguments will look familiar, a point about which we shall make more comment later.

4. We begin with a mild variant of a standard result from the Dirichlet-log-modular theory [W2, Lemma 5].

LEMMA 4. *Suppose I is an ideal of* A, $f \in L^\infty(\mu)$, *and f lies in the* $L^2(\mu)$ *closure of I. Then we can find a sequence* $\{f_n\}$ *in I such that*

$$\|f_n\| = \sup\{|f_n(x)| : x \in X\} \leqq \|f\|_\infty$$

and $f_n \to f$ *a.e.* (μ).

Proof. The proof is actually the same as Wermer's. One uses Theorem 1 instead of the Dirichlet property at a key point, and observes that if Wermer's f_n are in I, so are his h_v. Glicksberg has also observed that this generalization holds [G, remark after Theorem 2.1].

Briefly, assume (as we may) that $\|f\|_\infty = 1$ and let $\{k_n\}$ be a sequence in I with $\|f - k_n\|_2 \to 0$. Define $u_n = -\text{Log}^+ |k_n| \in C_R(X)$. One shows that $\int u_n \, d\mu \to 0$. Use Theorem 1 to pick $g_n \in A$ with $\text{Re}(g_n) \leqq u_n$, $\text{Im}\,\varphi(g_n) = 0$, and $\text{Re}\,\varphi(g_n) > \int u_n \, d\mu - 1/n$. Then $\|\exp(g_n)\| \leqq 1$ and $\varphi(\exp(g_n)) \to 1$, from which one shows that $\exp(g_n) \to 1$ in $L^2(\mu)$. Setting $f_n = k_n \exp(g_n) \in I$, it follows that $f_n \to f$ in $L^1(\mu)$. Since $\|f_n\| \leqq 1$, a subsequence of $\{f_n\}$ satisfies the conclusion of the lemma. □

LEMMA 5. *Suppose* $L \in (A_\varphi^n)^\perp$ *for a positive integer n, i.e., L is a continuous linear functional on A which annihilates* A_φ^n. *Let* λ *be a complex measure representing L and let* $\lambda = \lambda_a + \lambda_s$ *be its Lebesgue decomposition with respect to* μ. *Then* λ_a *represents L, i.e.,* λ_s *annihilates A.*

Proof. We use induction on n. The case $n = 1$ is the F. and M. Riesz theorem [H, Theorem 6.5]. Suppose $N > 1$ and we know the lemma for $1 \leqq n < N$. Let $L \in (A_\varphi^N)^\perp$ have the complex representing measure λ.

Let $g \in A_\varphi^{N-1}$ and define $L_1 \in A_\varphi^\perp$ by $L_1(f)=L(fg)$. If $d\lambda_1=g\,d\lambda$ then λ_1 represents L_1, so the case $n=1$ tells us that $(\lambda_1)_s$ annihilates A, i.e., $f\in A \Rightarrow \int fg\,d\lambda_s=\int f\,d(\lambda_1)_s=0$. Taking $f=1$ we obtain $\int g\,d\lambda_s=0$.

Thus λ_s annihilates A_φ^{N-1}, so by our induction assumption $\lambda_s=(\lambda_s)_s$ annihilates A. □

REMARK. A rather different and in some ways more satisfactory route to Lemma 5 is available. Ahern [A] has observed that, in considerably more generality than we need here, the F. and M. Riesz theorem can be made to follow from a lemma patterned after a theorem of Forelli [F, Theorem 1]. Glicksberg [G] has carried out this program in a form quite convenient for us: his proof of the F. and M. Riesz theorem [G, Theorem 1.1] from a Forelli-type lemma [G, Lemma 1.2] can be trivially modified to give our Lemma 5 for all n simultaneously.

We have instead used the proof above for two reasons. First, Hoffman's paper [H] is our basic text, and our proof seems to be the quickest route from Hoffman's paper to Lemma 5. Second, this (admittedly trivial) proof is evidently applicable to a perhaps much larger class of situations: we have a projection $\lambda \to \lambda_s$ satisfying certain conditions (the conclusion of the F. and M. Riesz theorem) relative to an algebra A and an ideal I, and we draw the same conclusion for the ideals I^n; it is possible to reformulate the entire affair in purely algebraic terms, and conclude that if a certain kind of projection "satisfies an F. and M. Riesz theorem" with respect to a suitable algebra A and ideal I, then it also "satisfies an F. and M. Riesz theorem" with respect to A and I^n.

$H^p(\mu)$ denotes the closure of A in $L^p(\mu)$, $1\leqq p<\infty$, and $H^\infty(\mu)=H^2(\mu)\cap L^\infty(\mu)$. Lemma 4 and bounded convergence show that $H^\infty(\mu)$ consists of the a.e. (μ) pointwise limits of bounded sequences in A, hence is a Banach algebra when endowed with a.e. (μ) pointwise operations and the L^∞ norm. Let $H_\varphi^p(\mu)=\{f\in H^p(\mu) : \int f\,d\mu=0\}$, $1\leqq p\leqq\infty$. In particular, if $1\leqq p<\infty$, then $H_\varphi^p(\mu)$ is the L^p closure of A_φ.

$\varphi' \in P_\varphi$ has a unique representing measure μ', and μ and μ' are mutually boundedly absolutely continuous. Thus the spaces $L^p(\mu)$ and $L^p(\mu')$ are identical as function spaces, as are $H^p(\mu)$ and $H^p(\mu')$, $1\leqq p\leqq\infty$, and the respective pairs of norms are equivalent.

For each $f\in L^1(\mu)$ we can therefore define a function $\hat f$ on P_φ by $\hat f(\varphi')=\int f\,d\mu'$. Clearly this agrees with the usual notion of $\hat f|P_\varphi$ if $f\in A$. Further, $f\to\hat f(\varphi')$ is a bounded linear functional on $L^1(\mu)$ (hence on $L^p(\mu)$, $1\leqq p\leqq\infty$) for each $\varphi'\in P_\varphi$.

LEMMA 6 (see [H, Theorem 5.1]). *$f\to\hat f(\varphi')$ is multiplicative on $H^2(\mu)$ for each $\phi'\in P_\varphi$, in the sense that if $f,g\in H^2(\mu)$ then $fg\in H^1(\mu)$ and $(fg)\hat{}(\varphi')=\hat f(\varphi')\cdot\hat g(\varphi')$. In particular, $f\to\hat f(\varphi')$ is multiplicative on $H^\infty(\mu)$.*

LEMMA 7. *If $L\in(A_\varphi^n)^\perp$ for a positive integer n, L extends to $L_1\in L^\infty(\mu)^*$ in such a way that $\|L_1\|=\|L\|$, L_1 annihilates $(H_\varphi^\infty(\mu))^n$, and L_1 is weakly continuous, i.e., if $\{f_j\}$ is a bounded sequence in $L^\infty(\mu)$ and $f_j\to g$ a.e. (μ) then $L_1(f_j)\to L_1(g)$.*

Proof. Select λ a complex measure representing L such that $\|\lambda\| = \|L\|$. Lemma 5 shows that $d\lambda = h\,d\mu$ for some $h \in L^1(\mu)$. Lemma 4 then can be used to see that $L_1(f) = \int f\,d\lambda$ will do. □

THEOREM 8. *Suppose $P_\varphi \neq \{\varphi\}$. Then there is a homeomorphism h of the open unit disc D onto P_φ (in the A^* metric topology) such that (D, h) is an analytic disc at φ. If h_1 and h_2 are two such functions then $h_1^{-1} \circ h_2$ is an analytic homeomorphism of D onto itself. Any such function h satisfies*:

(a) *$\hat{f} \circ h$ is analytic for each $f \in H^2(\mu)$.*

(b) *If $L \in (A_\varphi^n)^\perp$ for a positive integer n and L_1 is the extension of L to $L^\infty(\mu)$ guaranteed by Lemma 7, then $L_1 | H^\infty(\mu)$ has the form $L_1(f) = \sum_{k=0}^{n-1} a_k(d^k/dz^k)(\hat{f} \circ h)(0)$ for appropriate constants $a_0, \ldots, a_{n-1}$.*

Proof. Theorems 7.6 and 7.4 of [H] imply the existence of a $Z \in H_\varphi^\infty(\mu)$ with the following properties:

(i) $|Z| = 1$ a.e. (μ).

(ii) $\hat{Z}$ maps P_φ (in the A^* metric topology) homeomorphically onto D.

(iii) The function $f - \sum_{k=0}^{m} (\int \bar{Z}^k f\,d\mu) Z^k$ is in $Z^{m+1}H^2(\mu)$ whenever $f \in H^2(\mu)$ and m is a nonnegative integer.

(iv) $\varphi' \in P_\varphi, f \in H^2(\mu) \Rightarrow \hat{f}(\varphi') = \sum_{k=0}^{\infty} (\int \bar{Z}^k f\,d\mu) \hat{Z}(\varphi')^k$.

Then $h = (\hat{Z} | P_\varphi)^{-1}$ satisfies everything except (b), and we now verify (b). Define $a_k = L_1(Z^k)/k!$ for $0 \leqq k \leqq n-1$. Then $a_k(d^k/dz^k)(\hat{f} \circ h)(0) = (\int \bar{Z}^k f\,d\mu) L_1(Z^k)$ for $f \in H^2(\mu)$. If $f \in H^\infty(\mu)$ then (i) and (iii) show that $f - \sum_{k=0}^{n-1} (\int \bar{Z}^k f\,d\mu) Z^k$ is in $Z^n H^\infty(\mu) \subset (H_\varphi^\infty(\mu))^n$, so $L_1(f - \sum_{k=0}^{n-1} (\int \bar{Z}^k f\,d\mu) Z^k) = 0$ and therefore

$$L_1(f) = \sum_{k=0}^{n-1} \left(\int \bar{Z}^k f\,d\mu \right) L_1(Z^k) = \sum_{k=0}^{n-1} a_k (d^k/dz^k)(\hat{f} \circ h)(0).$$

This proves (b).

Now let h' be a second function mapping D homeomorphically and analytically onto P_φ. Then $h^{-1} \circ h'$ is a homeomorphism of D onto itself. Lemma 4 permits us to find a bounded sequence $\{f_j\}$ in A such that $f_j \to Z$ a.e. (μ). Then $\hat{f}_j | P_\varphi \to \hat{Z}$ pointwise, so $\hat{f}_j \circ h' \to \hat{Z} \circ h'$ pointwise. Since $\hat{f}_j \circ h'$ is analytic by hypothesis, so is $\hat{Z} \circ h' = h^{-1} \circ h'$. Thus $h^{-1} \circ h'$ is a diffeomorphism. This also implies that (a) and (b) hold for h', completing the proof. □

PROPOSITION 9. *Let Z be as in Theorem 8, $h = (\hat{Z} | P_\varphi)^{-1}$, n a positive integer. Define L on A by $L(f) = (d^{n-1}/dz^{n-1})(\hat{f} \circ h)(0)$. Then $L \in (A_\varphi^n)^\perp$, but $L \notin (A_\varphi^{n-1})^\perp$.*

Proof. Using the Cauchy integral representation for derivatives, $|L(f)| \leqq (n-1)!\,\|f\|$, so $L \in A^*$. $f \in A_\varphi \Rightarrow f = f - (\int \bar{Z}^0 f\,d\mu) Z^0 \Rightarrow f \in ZH^\infty(\mu)$ by (iii), so $f \in A_\varphi^n \Rightarrow f = Z^n g$ where $g \in H^\infty(\mu)$. Therefore $(\hat{f} \circ h)(z) = z^n \cdot (\hat{g} \circ h)(z)$ where $\hat{g} \circ h$ is analytic on D, so $L(f) = 0$. Therefore $L \in (A_\varphi^n)^\perp$.

On the other hand, Lemma 4 enables us to select a bounded sequence $\{f_j\}$ in A_φ such that $f_j \to Z$ a.e. (μ). Then $\hat{f}_j | P_\varphi \to \hat{Z}$ pointwise, so $\hat{f}_j \circ h \to \hat{Z} \circ h$ pointwise. It follows that $L(f_j^{n-1}) \to (n-1)!$ while $f_j^{n-1} \in A_\varphi^{n-1}$. Thus $L \notin (A_\varphi^{n-1})^\perp$. □

Theorem 8 and Proposition 9 prove that portion of Theorem 3 dealing with the case $P_\varphi \neq \{\varphi\}$. The case $P_\varphi = \{\varphi\}$ will now be covered by showing that if $A_\varphi \neq (A^2_\varphi)^-$ then there is a nontrivial analytic disc at φ.

Let V denote the closure in $L^2(\mu)$ of A^2_φ. Clearly $V \subset H^2_\varphi(\mu)$.

LEMMA 10. $A_\varphi \neq (A^2_\varphi)^- \Rightarrow H^2_\varphi(\mu) \neq V$.

Proof. Select f_0 in A_φ but not in $(A^2_\varphi)^-$. Then $f_0 \in H^2_\varphi(\mu)$.

We can find $L \in (A^2_\varphi)^\perp$ such that $L(f_0) \neq 0$. Suppose $f_0 \in V$. By Lemma 4 we can select a bounded sequence $\{f_j\}$ in A^2_φ such that $f_j \to f_0$ a.e. (μ). On the one hand $L(f_j) = 0$. On the other hand, Lemma 7 implies that L is weakly continuous, hence $L(f_j) \to L(f_0) \neq 0$, a contradiction. Thus $f_0 \notin V$. □

THEOREM 11. *If $A_\varphi \neq (A^2_\varphi)^-$ there exists $G \in H^\infty_\varphi(\mu)$ satisfying*

(i) $|G| = 1$ *a.e.* (μ).

(ii) *G spans $H^2_\varphi(\mu) \cap V^\perp$.*

(iii) $GH^2(\mu) = H^2_\varphi(\mu)$.

Proof. This will be a familar invariant subspace argument, and will really be the proof of the following more general theorem (see, e.g., [SW, Theorem 3.1]): If M is a singly invariant closed subspace of $L^2(\mu)$ (i.e., the closed linear span of $A_\varphi M$ is a proper subspace of M) then $M = GH^2(\mu)$ where $|G| = 1$ a.e. (μ).

By Lemma 10 we can select $G \in H^2_\varphi(\mu) \cap V^\perp$ such that $\|G\|_2 = 1$. We show that (i), (ii) and (iii) hold.

$$f \in A_\varphi \Rightarrow Gf \in V \Rightarrow Gf \perp G \Rightarrow \int f|G|^2\, d\mu = 0.$$

Thus $|G|^2\, d\mu$ represents φ, so uniqueness of μ gives (i).

If (ii) is false we can find orthonormal $G_1, G_2 \in H^2_\varphi(\mu) \cap V^\perp$. Let (a_1, a_2) be a pair of complex constants such that $|a_1|^2 + |a_2|^2 = 1$. Then $a_1G_1 + a_2G_2 \in H^2_\varphi(\mu) \cap V^\perp$ and $\|a_1G_1 + a_2G_2\|_2 = 1$, so again $|a_1G_1 + a_2G_2| = 1$ a.e. (μ). It is easily seen that this cannot hold simultaneously for all such pairs (a_1, a_2). Therefore (ii) must be true.

In view of (i) and Lemma 6, $GH^2(\mu)$ is a closed subspace of $H^2_\varphi(\mu)$. Suppose $g \in H^2_\varphi(\mu) \cap (GH^2(\mu))^\perp$. Then $f \in A \Rightarrow Gf \in GH^2(\mu) \Rightarrow Gf \perp g \Rightarrow \int \bar{f}\bar{G}g\, d\mu = 0$. On the other hand, $f \in A_\varphi \Rightarrow fg \in V \Rightarrow fg \perp G \Rightarrow \int f\bar{G}g\, d\mu = 0$. Thus $\bar{G}g\, d\mu$ annihilates $\bar{A} + A_\varphi = A + \bar{A}$, so by Theorem 6.7 of [H] $\bar{G}g = 0$ a.e. (μ). Because of (i), $g = 0$ a.e. (μ). Thus (iii) holds. □

LEMMA 12. *Suppose $A_\varphi \neq (A^2_\varphi)^-$ and G is as in Theorem 11. Then whenever $f \in H^2(\mu)$ and n is a nonnegative integer, we have $g_n \in G^{n+1}H^2(\mu)$ where*

$$g_n = f - \sum_{k=0}^{n} \left(\int \bar{G}^k f\, d\mu\right) G^k.$$

Proof. Induction on n. □

With hypotheses as in Lemma 12, for each $z \in D$ define a linear functional $\tilde{z}$ on

$L^2(\mu)$ by $\tilde{z}(f)=\sum_{n=0}^{\infty}(\int \bar{G}^n f\,d\mu)z^n$. $\tilde{z}$ is bounded (with norm at most $(1-|z|)^{-1}$) and for each $f\in L^2(\mu)$ the function $z\to\tilde{z}(f)$ is analytic on D.

THEOREM 13. *The map* $z\to\tilde{z}|A$ *is a nontrivial* (*in fact* 1-1) *analytic disc at* φ. *In particular,* $A_\varphi\neq(A_\varphi^2)^-\Rightarrow P_\varphi\neq\{\varphi\}$.

Proof. Using Lemma 12 it is easy to see that $\tilde{z}$ is multiplicative on $H^\infty(\mu)$, hence on A; since also $\tilde{z}(1)=1$, $\tilde{z}|A\in M_A$. Clearly the map is "analytic", and Theorem 11 leads to $\tilde{0}|A=\varphi$. Thus $z\to\tilde{z}|A$ is an analytic disc at φ. Finally, select a sequence $\{f_j\}\subset A$ such that $f_j\to G$ in $L^2(\mu)$. Then for each $z\in D$, $\tilde{z}(f_j)\to\tilde{z}(G)=z$. Thus the map is 1-1. □

REMARK. Lemma 12 and Theorem 13 are essentially the argument used by Wermer in [W] and repeated by Hoffman in [H] to put a disc in M_A. Theorem 13 completes the proof of Theorem 3.

5. In this section we use our characterization of the ideal $(A_\varphi^n)^-$ to see how certain behavior on X of functions in A can imply their belonging or at least being close to $(A_\varphi^n)^-$. Until further notice, we do not assume φ satisfies condition (U).

If $\varphi\in M_A$, a Jensen measure for φ on X is a measure μ of a total mass 1 such that the "Jensen inequality" $\log|\varphi(f)|\leqq\int\log|f|\,d\mu$ holds for all $f\in A$. Such a μ is easily seen to be an Arens-Singer measure for φ (i.e., $\log|\varphi(f)|=\int\log|f|\,d\mu$ for all invertible $f\in A$) and therefore a representing measure for φ (since $\operatorname{Re}(A)\subset\log|A^{-1}|$). Bishop has shown [B] that φ always has a Jensen measure on X.

LEMMA 14. *Suppose* μ *is a Jensen measure for* φ *and* E *is a Borel set in* X *such that* $\mu(E)>0$. *Suppose* $\{f_j\}$ *is a sequence in* A *and* $g\in L^1(\mu)$ *is such that* $|f_j|\leqq g$ *a.e.* (μ), *and suppose* $f_j\to 0$ *pointwise on* E. *Then* $\varphi(f_j)\to 0$.

Proof. If $\varepsilon>0$ is given, select $\delta>0$ so small that $(\log(\varepsilon)-\int g\,d\mu)/\log(\delta)<\mu(E)$ and $\delta<1$. If $E_j=\{x\in E: |f_j(x)|<\delta\}$ we can find J so large that $j\geqq J\Rightarrow\mu(E_j)>(\log(\varepsilon)-\int g\,d\mu)/\log(\delta)$. Then

$$\begin{aligned} j\geqq J\Rightarrow \log|\varphi(f_j)| &\leqq \int\log|f_j|\,d\mu \\ &= \int_{E_j}\log|f_j|\,d\mu+\int_{X-E_j}\log|f_j|\,d\mu \\ &\leqq \mu(E_j)\log(\delta)+\int g\,d\mu < \log(\varepsilon), \end{aligned}$$

so $|\varphi(f_j)|<\varepsilon$. □

LEMMA 15. *Let* U *and* V *be open subsets of the complex plane with respective coordinates* u *and* v. *Let* $\tau\colon U\to V$ *be analytic. Then for* $1\leqq k<\infty$ *and* $1\leqq l\leqq k$, *there exist polynomials* $Q_{k,l}$ *in* k *variables* $x_1,\dots,x_k$ *such that whenever* f *is an analytic function on* V, *we have*

$$(f\circ\tau)^{(k)}(u)=\sum_{l=1}^{k}Q_{k,l}(\tau'(u),\dots,\tau^{(k)}(u))f^{(l)}(\tau(u)),\quad 1\leqq k<\infty.$$

Proof. Induction on k. □

If $\{T_\alpha\}$ is a family of subspaces of A, a sequence $\{f_j\}$ in A will be said to converge to $\{T_\alpha\}$, written $f_j \to \{T_\alpha\}$, if $\lim_j (\sup_\alpha \inf\{\|f_j-g\| : g \in T_\alpha\})=0$. An easy application of the Hahn-Banach theorem shows that this is equivalent to the following: if S is the closed unit ball in A^*, then $f_j \to 0$ uniformly on $\bigcup_\alpha (S \cap T_\alpha^\perp)$ where f_j is interpreted as being in A^{**}.

THEOREM 16. *Suppose φ satisfies condition* (U) *and μ is a representing measure for φ. Suppose E is a Borel set in X and $\mu(E)>0$. Suppose $\{f_j\}$ is a bounded sequence in A such that $f_j \to 0$ pointwise on E. Then for each positive integer n,*

$$f_j \to \{A^n_{\varphi'} : \varphi' \in F\}$$

where F is any metrically compact subset of P_φ.

Proof. Let $\varphi' \in P_\varphi$ have representing measure μ'. By Theorem 2 φ' satisfies condition (U) and since φ' has a Jensen measure on X, μ' must be that Jensen measure. Lemma 14 then implies that $\varphi'(f_j) \to 0$. Thus $\hat{f}_j \to 0$ pointwise on P_φ.

If $P_\varphi=\{\varphi\}$ then Theorem 3 says $(A^n_\varphi)^-=A_\varphi$ and we are done.

Assume $P_\varphi \neq \{\varphi\}$ and let Z be as in Theorem 8, $h=(\hat{Z}|P_\varphi)^{-1}$. $\{\hat{f}_j \circ h\}$ is a bounded sequence of analytic functions on D such that $\hat{f}_j \circ h \to 0$ pointwise, hence for every nonnegative integer k, $(\hat{f}_j \circ h)^{(k)} \to 0$ uniformly on any compact subset of D.

We must show that $f_j \to 0$ uniformly on $\bigcup \{S \cap (A^n_{\varphi'})^\perp : \varphi' \in F\}$. We will accomplish this by finding a sequence of positive constants $\{c_k\}$ such that $L \in S \cap (A^n_{\varphi'})^\perp$ for some $\varphi' \in F$ implies $|L(f_j)| \leqq \sum_{k=0}^{n-1} c_k \|(f_j \circ h)^{(k)}|\hat{Z}(F)\|$. Since $\hat{Z}(F)$ is a compact subset of D, $(\hat{f}_j \circ h)^{(k)} \to 0$ uniformly on $\hat{Z}(F)$, and the theorem will be proved. We define the c_k inductively by $c_0=1$ and $c_k=(1/k!)+\sum_{s=0}^{k-1} c_s/(k-s)!$ for $k>0$.

Suppose $\varphi' \in F$. Let Z' be constructed for φ' as in Theorem 8 and set $h'=(\hat{Z}'|P_\varphi)^{-1}$. Define an analytic function $\tau=\hat{Z} \circ h' : D \to D$. ((a) in Theorem 8 shows that τ is indeed analytic.)

Let $L \in S \cap (A^n_{\varphi'})^\perp$ be given and let L_1 be the extension of L guaranteed by Lemma 7. By Theorem 8 $L_1|H^\infty(\mu)$ has the form $L_1(f)=\sum_{k=0}^{n-1} a_k(\hat{f} \circ h')^{(k)}(0)$. Lemma 15 then implies that $L_1|H^\infty(\mu)$ has the form $L_1(f)=\sum_{k=0}^{n-1} b_k(\hat{f} \circ h)^{(k)}(\hat{Z}(\varphi'))$. We will be done if $|b_k| \leqq c_k$ for all k. This we show by induction on k.

Observe that $|L_1(Z^k)| \leqq \|L_1\|=\|L\| \leqq 1$ for all $k \geqq 0$. Applied to the case $k=0$ this gives $|b_0| \leqq 1=c_0$.

Suppose $1 \leqq K \leqq n-1$ and $|b_k| \leqq c_k$ for $0 \leqq k<K$. Then

$$\begin{aligned} 1 \geqq |L_1(Z^K)| &= \left|\sum_{s=0}^{n-1} b_s(\hat{Z}^K \circ h)^{(s)}(\hat{Z}(\varphi'))\right| \\ &= \left|\sum_{s=0}^{K} K!\,b_s(\hat{Z}(\varphi'))^{K-s}/(K-s)!\right| \\ &= K!\left|b_K+\sum_{s=0}^{K-1} b_s(\hat{Z}(\varphi'))^{K-s}/(K-s)!\right| \end{aligned}$$

so that

$$|b_K| \leqq \frac{1}{K!} + \left| \sum_{s=0}^{K-1} b_s(\hat{Z}(\varphi'))^{K-s}/(K-s)! \right| \leqq \frac{1}{K!} + \sum_{s=0}^{K-1} c_s/(K-s)! = c_K. \quad \square$$

COROLLARY 17. *Let* φ, μ *and* E *be as in Theorem* 16. *Suppose* $f \in A$ *and* $f|E=0$. *Then*

$$f \in \bigcap \{(A^n_{\varphi'})^- : \varphi' \in P_\varphi, 1 \leqq n < \infty\}.$$

REFERENCES

[A] P. R. Ahern, *On the generalized F. and M. Riesz theorem*, Pacific J. Math. **15** (1965), 373–376.

[B] E. Bishop, *Holomorphic completions, analytic continuation and the interpolation of semi-norms*, Ann. of Math. **78** (1963), 468–500.

[F] F. Forelli, *Analytic measures*, Pacific J. Math. **13** (1963), 571–578.

[G] I. Glicksberg, *The abstract F. and M. Riesz theorem*, J. Functional Analysis **1** (1967), 109–122.

[H] K. Hoffman, *Analytic functions and logmodular Banach algebras*, Acta Math. **108** (1962), 271–317.

[L] G. Lumer, *Analytic functions and Dirichlet problem*, Bull. Amer. Math. Soc. **70** (1964), 98–104.

[S] S. J. Sidney, *Powers of maximal ideals in function algebras*, Thesis, Harvard Univ., Cambridge, Mass., 1966.

[SW] T. P. Srinivasan and J. Wang, "Weak*-Dirichlet algebras," in *Function algebras*, edited by F. T. Birtel, pp. 216–249, Scott, Foresman and Company, Chicago, Ill., 1966.

[W1] J. Wermer, *Dirichlet algebras*, Duke Math. J. **27** (1960), 373–382.

[W2] ———, *Seminar über Funktionen-Algebren*, Lecture Notes in Mathematics, No. 1, Springer-Verlag, Berlin, 1964.

HARVARD UNIVERSITY,
CAMBRIDGE, MASSACHUSETTS
YALE UNIVERSITY,
NEW HAVEN, CONNECTICUT